Interlacing Log-concavity of the Boros-Moll Polynomials

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Abstract. We introduce the notion of interlacing log-concavity of a polynomial sequence $\{P_m(x)\}_{m\geq 0}$, where $P_m(x)$ is a polynomial of degree m with positive coefficients $a_i(m)$. This sequence of polynomials is said to be interlacing log-concave if the ratios of consecutive coefficients of $P_m(x)$ interlace the ratios of consecutive coefficients of $P_{m+1}(x)$ for any $m \geq 0$. Interlacing log-concavity is stronger than the log-concavity. We show that the Boros-Moll polynomials are interlacing log-concave. Furthermore we give a sufficient condition for interlacing log-concavity which implies that some classical combinatorial polynomials are interlacing log-concave.

Keywords: interlacing log-concavity, log-concavity, Boros-Moll polynomial

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1 Introduction

In this paper, we introduce the notion of interlacing log-concavity of a polynomial sequence $\{P_m(x)\}_{m\geq 0}$, which is stronger than the log-concavity of the polynomials $P_m(x)$. We shall show that the Boros-Moll polynomials are interlacing log-concave.

For a sequence polynomials $\{P_m(x)\}$, let

$$P_m(x) = \sum_{i=0}^{m} a_i(m)x^m,$$

and let $r_i(m) = a_i(m)/a_{i+1}(m)$. We say that the polynomials $P_m(x)$ are interlacing log-concave if the ratios $r_i(m)$ interlace the ratios $r_i(m+1)$, that is,

$$r_0(m+1) \leq r_0(m) \leq r_1(m+1) \leq r_1(m) \leq \cdots \leq r_{m-1}(m+1) \leq r_{m-1}(m) \leq r_m(m+1). \tag{1.1}$$

Recall that a sequence $\{a_i\}_{0\leq i\leq m}$ of positive numbers is said to be log-concave if

$$\frac{a_0}{a_1} \leq \frac{a_1}{a_2} \leq \cdots \leq \frac{a_{m-1}}{a_m}.$$

It is clear that the interlacing log-concavity implies the log-concavity.

For the background on the Boros-Moll polynomials; see [1–6, 10]. From now on, we shall use $P_m(a)$ to denote the Boros-Moll polynomial given by

$$P_m(x) = \sum_{j,k} \binom{2m+1}{2j}\binom{m-j}{k}\binom{2k+2j}{k+j}\frac{(x+1)^j(x-1)^k}{2^{3(k+j)}}. \tag{1.2}$$

Boros and Moll [2] derived the following formula for the coefficient $d_i(m)$ of x^i in $P_m(x)$,

$$d_i(m) = 2^{-2m}\sum_{k=i}^{m} 2^k\binom{2m-2k}{m-k}\binom{m+k}{k}\binom{k}{i}. \tag{1.3}$$

Boros and Moll [3] proved that the sequence $\{d_i(m)\}_{0\leq i\leq m}$ is unimodal and the maximum element appears in the middle. In other words,

$$d_0(m) < d_1(m) < \cdots < d_{[\frac{m}{2}]}(m) > d_{[\frac{m}{2}]-1}(m) > \cdots > d_m(m). \tag{1.4}$$

Moll [10] conjectured $P_m(x)$ is log-concave for any m. Kauers and Paule [9] confirmed this conjecture based on recurrence relations found by a computer algebra approach. Chen and Xia [7] showed that the sequence $\{d_i(m)\}_{0\leq i\leq m}$ satisfies the ratio monotone property which implies the log-concavity and the spiral property. Chen and Gu showed that for any m, $P_m(x)$ is reverse ultra log-concave [8].

The main result of this paper is to show that the Boros-Moll polynomials are interlacing log-concave. We also give a sufficient condition for the interlacing log-concavity from which we see that several classical combinatorial polynomials are interlacing log-concave.

2 The interlacing log-concavity of $d_i(m)$

In this section, we show that for $m \geq 2$, the the Boros-Moll polynomials $P_m(x)$ are interlacing log-concave. More precisely, we have

Theorem 2.1. *For $m \geq 2$ and $0 \leq i \leq m$, we have*

$$d_i(m)d_{i+1}(m+1) > d_{i+1}(m)d_i(m+1) \tag{2.1}$$

and

$$d_i(m)d_i(m+1) > d_{i-1}(m)d_{i+1}(m+1). \tag{2.2}$$

The proof relies on the following recurrence relations derived by Kauers and Paule [9]. In fact, they found four recurrence relations for the Boros-Moll sequence $\{d_i(m)\}_{0\leq i\leq m}$:

$$d_i(m+1) = \frac{m+i}{m+1}d_{i-1}(m) + \frac{(4m+2i+3)}{2(m+1)}d_i(m), \qquad 0 \leq i \leq m+1, \tag{2.3}$$

$$d_i(m+1) = \frac{(4m-2i+3)(m+i+1)}{2(m+1)(m+1-i)} d_i(m) - \frac{i(i+1)}{(m+1)(m+1-i)} d_{i+1}(m), \qquad 0 \le i \le m, \tag{2.4}$$

$$d_i(m+2) = \frac{-4i^2+8m^2+24m+19}{2(m+2-i)(m+2)} d_i(m+1) - \frac{(m+i+1)(4m+3)(4m+5)}{4(m+2-i)(m+1)(m+2)} d_i(m), \qquad 0 \le i \le m+1, \tag{2.5}$$

and for $0 \le i \le m+1$,

$$(m+2-i)(m+i-1)d_{i-2}(m) - (i-1)(2m+1)d_{i-1}(m) + i(i-1)d_i(m) = 0. \tag{2.6}$$

Note that Moll [11] also has independently derived the recurrence relation (2.6) from which the other three relations can be deduced.

To prove (2.1), we give the following lemma.

Lemma 2.2. *Let $m \ge 2$ be an integer. For $0 \le i \le m-2$, we have*

$$\frac{d_i(m)}{d_{i+1}(m)} < \frac{(4m+2i+3)d_{i+1}(m)}{(4m+2i+7)d_{i+2}(m)}. \tag{2.7}$$

Proof. We proceed by induction on m. It is easy to check that the theorem is valid for $m = 2$. Assume that the result is true for n, that is, for $0 \le i \le n-2$,

$$\frac{d_i(n)}{d_{i+1}(n)} < \frac{(4n+2i+3)d_{i+1}(n)}{(4n+2i+7)d_{i+2}(n)}. \tag{2.8}$$

We aim to show that (2.7) holds for $n+1$, that is, for $0 \le i \le n-1$,

$$\frac{d_i(n+1)}{d_{i+1}(n+1)} < \frac{(4n+2i+7)d_{i+1}(n+1)}{(4n+2i+11)d_{i+2}(n+1)}. \tag{2.9}$$

From the recurrence relation (2.3), we can verify that for $0 \le i \le n-1$,

$$\begin{aligned}
&(2i+4n+7)d_{i+1}^2(n+1) - (2i+4n+11)d_i(n+1)d_{i+2}(n+1) \\
&= (2i+4n+7)\left(\frac{i+n+1}{n+1}d_i(n) + \frac{2i+4n+5}{2(n+1)}d_{i+1}(n)\right)^2 \\
&\quad - (2i+4n+11)\left(\frac{i+n+2}{n+1}d_{i+1}(n) + \frac{2i+4n+7}{2(n+1)}d_{i+2}(n)\right) \\
&\quad \times \left(\frac{n+i}{n+1}d_{i-1}(n) + \frac{2i+4n+3}{2(n+1)}d_i(n)\right)
\end{aligned}$$

$$= \frac{A_1(n,i)+A_2(n,i)+A_3(n,i)}{4(n+1)^2},$$

where $A_1(n,i)$, $A_2(n,i)$ and $A_3(n,i)$ are given by

$$\begin{aligned}
A_1(n,i) &= 4(2i+4n+7)(i+n+1)^2 d_i^2(n) \\
&\quad - 4(n+i)(2i+4n+11)(i+n+2)d_{i+1}(n)d_{i-1}(n), \\
A_2(n,i) &= (2i+4n+7)(2i+4n+5)^2 d_{i+1}^2(n) \\
&\quad - (2i+4n+3)(2i+4n+11)(2i+4n+7)d_i(m)d_{i+2}(n), \\
A_3(n,i) &= (8i^3+40i^2+58i+32n^3+42n+80n^2+120ni+40i^2n+64n^2i+8) \\
&\quad \cdot d_{i+1}(n)d_i(n) - 2(n+i)(2i+4n+11)(2i+4n+7)d_{i+2}(n)d_{i-1}(n).
\end{aligned}$$

We claim that $A_1(n,i)$, $A_2(n,i)$ and $A_3(n,i)$ are positive for $0 \le i \le n-2$. By the inductive hypothesis (2.8), we find that for $0 \le i \le n-2$,

$$\begin{aligned}
A_1(n,i) > &\, 4(2i+4n+7)(i+n+1)^2 d_i^2(n) \\
&- 4(n+i)(2i+4n+11)(i+n+2)\frac{(4n+2i+1)}{(4n+2i+5)}d_i^2(n) \\
= &\, 4\frac{35+96n+72i+64ni+40n^2+28i^2}{2i+4n+5}d_i^2(n),
\end{aligned}$$

which is positive. From (2.8) it follows that for $0 \le i \le n-2$,

$$\begin{aligned}
A_2(n,i) > &\, (2i+4n+7)(2i+4n+5)^2 d_{i+1}^2(n) \\
&- (2i+4n+3)(2i+4n+11)(2i+4n+7)\frac{(4n+2i+3)}{(4n+2i+7)}d_{i+1}^2(n) \\
= &\, (40i+80n+76)d_{i+1}^2(n),
\end{aligned}$$

which is also positive. By the inductive hypothesis (2.8), we see that for $0 \le i \le n-2$,

$$d_i(n)d_{i+1}(n) > \frac{(2i+4n+5)(2i+4n+7)}{(2i+4n+3)(2i+4n+1)}d_{i-1}(n)d_{i+2}(n). \tag{2.10}$$

Because of (2.10), we see that

$$\begin{aligned}
A_3(n,i) > &(8i^3+40i^2+58i+32n^3+42n+80n^2+120ni+40i^2n+64n^2i+8)d_{i+1}(n)d_i(n) \\
&- 2(n+i)(2i+4n+11)(2i+4n+7)\frac{(4n+2i+3)(4n+2i+1)}{(4n+2i+5)(4n+2i+7)}d_{i+1}(n)d_i(n) \\
= &8\frac{5+22n+30i+44ni+24n^2+16i^2}{2i+4n+5}d_{i+1}(n)d_i(n),
\end{aligned}$$

which is still positive for $0 \le i \le n-2$. Hence we deduce the inequality (2.9) for $0 \le i \le n-2$. It remains to check that (2.9) is true for $i = n-1$, that is,

$$\frac{d_{n-1}(n+1)}{d_n(n+1)} < \frac{(6n+5)d_n(n+1)}{(6n+9)d_{n+1}(n+1)}. \tag{2.11}$$

In view of (1.3), we get

$$d_n(n+1) = 2^{-n-2}(2n+3)\binom{2n+2}{n+1}, \tag{2.12}$$

$$d_{n+1}(n+1) = \frac{1}{2^{n+1}}\binom{2n+2}{n+1}. \tag{2.13}$$

$$d_n(n+2) = \frac{(n+1)(4n^2+18n+21)}{2^{n+4}(2n+3)}\binom{2n+4}{n+2}. \tag{2.14}$$

Consequently,

$$\frac{d_{n-1}(n+1)}{d_n(n+1)} = \frac{n(4n^2+10n+7)}{2(2n+1)(2n+3)} < \frac{(2n+3)(6n+5)}{2(6n+9)} = \frac{(6n+5)d_n(n+1)}{(6n+9)d_{n+1}(n+1)}.$$

This completes the proof. ∎

We now proceed to give a proof of (2.1). In fact we shall prove a stronger inequality.

Lemma 2.3. *Let $m \ge 2$ be a positive integer. For $0 \le i \le m-1$, we have*

$$\frac{d_i(m)}{d_{i+1}(m)} > \frac{(2i+4m+5)d_i(m+1)}{(2i+4m+3)d_{i+1}(m+1)}. \tag{2.15}$$

Proof. By Lemma 2.2, we have for $0 \le i \le m-1$,

$$d_i^2(m) > \frac{2i+4m+5}{2i+4m+1}d_{i-1}(m)d_{i+1}(m). \tag{2.16}$$

From (2.16) and the recurrence relation (2.3), we find that for $0 \le i \le m-1$,

$$\begin{aligned}
&d_{i+1}(m+1)d_i(m) - \frac{2i+4m+5}{2i+4m+3}d_{i+1}(m)d_i(m+1)\\
&\quad= \frac{2i+4m+5}{2(m+1)}d_{i+1}(m)d_i(m) + \frac{i+m+1}{m+1}d_i(m)^2\\
&\qquad - \frac{2i+4m+5}{2i+4m+3}\left(\frac{2i+4m+3}{2(m+1)}d_i(m)d_{i+1}(m) + \frac{i+m}{m+1}d_{i-1}(m)d_{i+1}(m)\right)\\
&\quad= \frac{i+m+1}{m+1}d_i^2(m) - \frac{(4m+2i+5)(m+i)}{(4m+2i+3)(m+1)}d_{i-1}(m)d_{i+1}(m)
\end{aligned}$$

$$> \left(\frac{m+1+i}{m+1} - \frac{(4m+2i+1)(m+i)}{(4m+2i+3)(m+1)} \right) d_i^2(m)$$
$$= \frac{6m+4i+3}{(4m+2i+3)(m+1)} d_i^2(m),$$

which is positive. This yields (2.15), and hence the proof is complete. ∎

Let us turn to the proof of (2.2).

Proof of (2.2). We proceed by induction on m. Clearly, the (2.2) holds for $m = 2$. We assume that it is true for $n \geq 2$, that is, for $0 \leq i \leq n-1$,

$$\frac{d_i(n)}{d_{i+1}(n)} < \frac{d_{i+1}(n+1)}{d_{i+2}(n+1)}. \tag{2.17}$$

It will be shown that the theorem holds for $n+1$, that is, for $0 \leq i \leq n$,

$$\frac{d_i(n+1)}{d_{i+1}(n+1)} < \frac{d_{i+1}(n+2)}{d_{i+2}(n+2)}. \tag{2.18}$$

From the unimodality (1.4), it follows that $d_i(n+1) < d_{i+1}(n+1)$ for $0 \leq i \leq \left[\frac{n+1}{2}\right] - 1$ and $d_i(n+1) > d_{i+1}(n+1)$ for $\left[\frac{n+1}{2}\right] \leq i \leq n$. From the recurrence relation (2.3), we find that for $0 \leq i \leq \left[\frac{n+1}{2}\right] - 1$,

$$\begin{aligned}
&d_{i+1}(n+1)d_{i+1}(n+2) - d_{i+2}(n+2)d_i(n+1) \\
&\quad = \frac{2i+4n+9}{2(n+2)} d_{i+1}^2(n+1) + \frac{i+n+2}{n+2} d_i(n+1)d_{i+1}(n+1) \\
&\qquad - \frac{2i+4n+11}{2(n+2)} d_i(n+1)d_{i+2}(n+1) - \frac{i+n+3}{n+2} d_i(n+1)d_{i+1}(n+1) \\
&\quad = \frac{2i+4n+9}{2(n+2)} d_{i+1}^2(n+1) - \frac{2i+4n+11}{2(n+2)} d_i(n+1)d_{i+2}(n+1) \\
&\qquad - \frac{1}{n+2} d_i(n+1)d_{i+1}(n+1) \\
&\quad > \frac{2i+4n+7}{2(n+2)} d_{i+1}^2(n+1) - \frac{2i+4n+11}{2(n+2)} d_i(n+1)d_{i+2}(n+1),
\end{aligned}$$

which is positive by Lemma 2.2. It follows that for $0 \leq i \leq \left[\frac{n+1}{2}\right] - 1$,

$$d_{i+1}(n+1)d_{i+1}(n+2) - d_{i+2}(n+2)d_i(n+1) > 0. \tag{2.19}$$

In other words, (2.2) is valid for $0 \leq i \leq \left[\frac{n+1}{2}\right] - 1$.

We now consider the case $\left[\frac{n+1}{2}\right] \leq i \leq n-1$. From the recurrence relations (2.3) and (2.4), it follows that for $\left[\frac{n+1}{2}\right] \leq i \leq n-1$,

$$d_{i+1}(n+2)d_{i+1}(n+1) - d_{i+2}(n+2)d_i(n+1)$$

$$
\begin{aligned}
&= \left(\frac{(4n-2i+5)(n+i+3)}{2(n+2)(n+1-i)} d_{i+1}(n+1) - \frac{(i+1)(i+2)}{(n+2)(n+1-i)} d_{i+2}(n+1)\right) \\
&\quad \times \left(\frac{n+1+i}{n+1} d_i(n) + \frac{4n+2i+5}{2(n+1)} d_{i+1}(n)\right) \\
&\quad - \left(\frac{n+3+i}{n+2} d_{i+1}(n+1) + \frac{4n+2i+11}{2(n+2)} d_{i+2}(n+1)\right) \\
&\qquad \times \left(\frac{(4n-2i+3)(n+i+1)}{2(n+1)(n+1-i)} d_i(n) - \frac{i(i+1)}{(n+1)(n+1-i)} d_{i+1}(n)\right) \\
&= B_1(n,i) d_{i+1}(n+1) d_i(n) + B_2(n,i) d_{i+1}(n+1) d_{i+1}(n) \\
&\qquad + B_3(n,i) d_{i+2}(n+1) d_i(n) + B_4(n,i) d_{i+2}(n+1) d_{i+1}(n),
\end{aligned}
$$

where $B_1(n,i)$, $B_2(n,i)$, $B_3(n,i)$ and $B_4(n,i)$ are given by

$$
B_1(n,i) = \frac{(n+i+3)(n+1+i)}{(n+2)(n+1-i)(n+1)}, \tag{2.20}
$$

$$
B_2(n,i) = \frac{(n+i+3)(16n^2+40n+25+4i)}{4(n+2)(n+1-i)(n+1)}, \tag{2.21}
$$

$$
B_3(n,i) = -\frac{(n+1+i)(41+16n^2+56n-4i)}{4(n+2)(n+1-i)(n+1)}, \tag{2.22}
$$

$$
B_4(n,i) = -\frac{(i+1)(4n+5-i)}{(n+2)(n+1-i)(n+1)}. \tag{2.23}
$$

Since $\left[\frac{n+1}{2}\right] \le i \le n-1$, it is clear from (1.4) that $d_{i+1}(n+1) > d_{i+2}(n+1)$ and $d_i(n) > d_{i+1}(n)$. Thus we get

$$
d_{i+1}(n+1) d_i(n) > d_{i+1}(n+1) d_{i+1}(n), \tag{2.24}
$$

$$
d_{i+1}(n+1) d_{i+1}(n) > d_{i+2}(n+1) d_{i+1}(n). \tag{2.25}
$$

Observe that $B_1(n,i)$, $B_2(n,i)$ are positive and $B_3(n,i)$, $B_4(n,i)$ are negative. By the inductive hypothesis (2.17), (2.24) and (2.25), we deduce that for $\left[\frac{n+1}{2}\right] \le i \le n-1$,

$$
\begin{aligned}
&d_{i+1}(n+2) d_{i+1}(n+1) - d_{i+2}(n+2) d_i(n+1) \\
&\qquad > \left(B_1(n,i) + B_2(n,i) + B_3(n,i) + B_4(n,i)\right) d_{i+1}(n+1) d_{i+1}(n) \\
&\qquad = \frac{24n+10n^2-8ni+8i^2+13}{2(n+2)(n+1-i)(n+1)} d_{i+1}(n+1) d_{i+1}(n) > 0.
\end{aligned} \tag{2.26}
$$

From the inequalities (2.19) and (2.26), it can be seen that (2.18) holds for $0 \le i \le n-1$.

We still are left with case $i = n$, that is,

$$
\frac{d_n(n+1)}{d_{n+1}(n+1)} < \frac{d_{n+1}(n+2)}{d_{n+2}(n+2)}. \tag{2.27}
$$

Applying (2.6) with $i = n+2$, we find that

$$\frac{d_n(n+1)}{d_{n+1}(n+1)} = \frac{2n+3}{2} < \frac{2n+5}{2} = \frac{d_{n+1}(n+2)}{d_{n+2}(n+2)},$$

as desired. This completes the proof. ∎

3 Examples of interlacing log-concave polynomials

Many combinatorial polynomials with only real zeros admit triangular relations on their coefficients. The log-concavity of polynomials of this kind have been extensively studied. We show that several classical polynomials that are interlacing log-concave. To this end, we give a criterion for interlacing log-concavity based on triangular relations on the coefficients.

Theorem 3.1. *Suppose that for any $n \geq 0$,*

$$G_n(x) = \sum_{k=0}^{n} T(n,k)x^k$$

is a polynomial of degree n which has only real zeros, and suppose that the coefficients $T(n,k)$ satisfy a recurrence relation of the following triangular form

$$T(n,k) = f(n,k)T(n-1,k) + g(n,k)T(n-1,k-1).$$

If

$$\frac{(n-k)k}{(n-k+1)(k+1)} f(n+1,k+1) \leq f(n+1,k) \leq f(n+1,k+1) \tag{3.1}$$

and

$$g(n+1,k+1) \leq g(n+1,k) \leq \frac{(n-k+1)(k+1)}{(n-k)k} g(n+1,k+1), \tag{3.2}$$

then the polynomials $G_n(x)$ are interlacing log-concave.

Proof. Given the condition that $G_n(x)$ has only real zeros, by Newton's inequality, we have

$$k(n-k)T(n,k)^2 \geq (k+1)(n-k+1)T(n,k-1)T(n,k+1).$$

Hence

$$\begin{aligned} &T(n,k)T(n+1,k+1) - T(n+1,k)T(n,k+1) \\ &\quad = f(n+1,k+1)T(n,k)T(n,k+1) + g(n+1,k+1)T(n,k)^2 \\ &\qquad - f(n+1,k)T(n,k)T(n,k+1) - g(n+1,k)T(n,k-1)T(n,k+1) \end{aligned}$$

$$
\begin{aligned}
&\geq \left(f(n+1,k+1) - f(n+1,k)\right) T(n,k)T(n,k+1) \\
&\qquad + \left(\frac{(n-k+1)(k+1)}{(n-k)k} g(n+1,k+1) - g(n+1,k)\right) T(n,k-1)T(n,k+1),
\end{aligned}
$$

which is positive by (3.1) and (3.2). It follows that

$$
\frac{T(n,k)}{T(n,k+1)} \geq \frac{T(n+1,k)}{T(n+1,k+1)}. \tag{3.3}
$$

On the other hand, we have

$$
\begin{aligned}
&T(n,k+1)T(n+1,k+1) - T(n,k)T(n+1,k+2) \\
&\quad = f(n+1,k+1)T(n,k+1)^2 + g(n+1,k+1)T(n,k)T(n,k+1) \\
&\qquad\quad - f(n+1,k+2)T(n,k)T(n,k+2) - g(n+1,k+2)T(n,k+1)T(n,k) \\
&\quad \geq \left(f(n+1,k+1) - \frac{(n-k-1)(k+1)}{(n-k)(k+2)} f(n+1,k+2)\right) T(n,k+1)^2 \\
&\qquad\quad + (g(n+1,k+1) - g(n+1,k+2))T(n,k+1)T(n,k).
\end{aligned}
$$

Invoking (3.1) and (3.2), we get

$$
\frac{T(n,k)}{T(n,k+1)} \leq \frac{T(n+1,k+1)}{T(n+1,k+2)}. \tag{3.4}
$$

Hence the proof is complete by combining (3.3) and (3.4). ∎

Theorem 3.1 we can show that many combinatorial polynomials which have only real zeros are interlacing log-concave. For example, the polynomials $(x+1)^n$, $x(x+1)\cdots(x+n-1)$, the Bell polynomials, and the Whitney polynomials

$$
W_{m,n}(x) = \sum_{k=1}^{n} W_m(n,k)x^k,
$$

where m is fixed nonnegative integer and the coefficients $W_m(n,k)$ satisfy the recurrence relation

$$
W_m(n,k) = (1+mk)W_m(n-1,k) + W_m(n-1,k-1).
$$

To conclude, we remark that numerical evidence suggests that the Boros-Moll polynomials possess higher order interlacing log-concavity in the spirit of the infinite-log-concavity as introduced by Moll [10].

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